THE RELATIONSHIP BETWEEN THE BEVERIDGE-NELSON DECOMPOSITION AND EXPONENTIAL SMOOTHING

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Abstract

In this article, two parallel decompositions of an ARIMA model are presented. Both of them are based on a partial fraction expansion of the model and can incorporate complex seasonal patterns. The first one coincides with the well known Beveridge–Nelson decomposition. The other constitutes an innovations form of the Beveridge–Nelson decomposition and coincides with the innovations form of many of the usual additive exponential smoothing models. It is shown that specifying complex models based on the Beveridge–Nelson decomposition and using its innovations form may provide a useful tool for forecasting. It is also shown that the Beveridge–Nelson decomposition is adequate for concurrent estimation of the unobserved components, but that multiple source of error models are more appropriate if estimation of the components based on the whole sample is required.

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1 Introduction

In this article, two partial fraction expansions of an ARIMA model are presented. They are based on what is known in electrical engineering as parallel decompositions of rational transfer functions of digital filters. The first decomposition coincides with the one proposed by Beveridge and Nelson (1981), henceforth BN, that has attracted considerable attention in the applied macroeconomics literature, but also generalizes it to seasonal models. The second one corresponds to the innovations form of the BN decomposition.

The two decompositions are analyzed using both state space and polynomial methods. It is shown that most of the usual additive exponential smoothing models or some generalizations of them are in fact BN decompositions of ARIMA models or generalizations of it. This fact seems to have passed unnoticed in the literature, although the link between single source of error (SSOE) state space models and exponential smoothing has been recognized (Hyndman et al., 2002) and used for some time (e.g., De Livera, Hyndman, and Snyder, 2011). It is also shown that these SSOE models are in fact innovations state space models corresponding to the BN decomposition that defines the model.

Based on the generalized BN decomposition, a specification of state space models with complex seasonal patterns is proposed that can handle hourly, daily or weekly data, that may have integer or even non integer seasonal periods. The state space innovations form of this BN decomposition is useful for model estimation and forecasting. However, these models present some differences with respect to the models proposed by De Livera et al. (2011) that may affect forecasting efficacy.

It is shown in the article that the generalized BN decomposition can be useful for the concurrent estimation of unobserved components, but that multiple source of error (MSOE) models are needed if one is interested in components estimated using the whole sample. The link between MSOE and SSOE estimators is also analyzed. It is proved that, when there is no correlation among the components in the MSOE model, the concurrent estimators coincide with the ones obtained using the SSOE model given by the BN decomposition.

The remainder of the article is organized as follows. In Section 2, the two parallel decompositions of an ARIMA model are presented. These two decompositions are analyzed using polynomial and state space methods. The connection with exponential smoothing is studied. A specification of models with complex seasonal patterns is proposed that is based on the BN decomposition, and its innovations state space form is used for model estimation and forecasting. Finally, signal extraction using the BN decomposition and MSOE models is analyzed and the results obtained with both approaches compared. An application to some real time series concludes in Section 3.

2 Two Parallel Decompositions of an ARIMA Model

In this section, we will consider what is known in digital filtering as parallel decompositions of rational transfer functions of digital filters (see Oppenheim and Schafer, 2010, pp. 390–395). These decompositions are based on partial fraction expansions of the transfer functions. Since a time series following an ARIMA model can be considered as the result of applying a rational filter to a white noise sequence, parallel decompositions can also be useful in time series analysis.

Suppose a time series $\{y_t\}$, $t = 1, \ldots, N$, that follows a multiplicative seasonal ARIMA model, i.e.

$$\phi(B)\Phi(B^n)(\nabla^d \nabla_n^D y_t - \mu) = \theta(B)\Theta(B^n)a_t, \tag{1}$$

where μ is the mean of the differenced series, B is the backshift operator, $By_t = y_{t-1}$, n is the number of seasons, $d = 0, 1, 2$, $D = 0, 1$, $\nabla = 1 - B$ is a regular difference and $\nabla_n = 1 - B^n$ is a seasonal difference. Our aim in this section is to decompose model (1) using two partial fraction expansions. This can be done using both polynomial and state space methods. However, it is to be emphasized that the developments of this and the following section are valid for any kind of ARIMA model, multiplicative seasonal or not. This means that we may consider general models with complex patterns of seasonality, like for example

$$\phi(B)\left[\nabla^d \prod_{i=1}^{N}(1 + B + B^2 + \cdots + B^{n_i - 1})y_t - \mu\right] = \theta(B)a_t, \tag{2}$$

where $n_1, \ldots, n_N$ denote the seasonal periods and the polynomials $\phi(z)$ and $\theta(z)$ have all their roots outside the unit circle but are otherwise unrestricted, or even models with seasonal patterns with non integer periods as we will see later.

The following lemma, that we give without proof, is an immediate consequence of the partial fraction expansion studied in algebra and will useful later. See also Lemma 1 of Gómez and Breitung (1999, p. 528) and the results contained therein.

Lemma 1 *Let the general ARIMA model $\nabla^d\phi(B)y_t = \theta(B)a_t$, where the roots of $\phi(z)$ are simple and on or outside of the unit circle and $\phi(z)$ and $\theta(z)$ have degrees p and q, respectively. Then, the following partial fraction decomposition holds*

$$\frac{\theta(z)}{(1-z)^d\phi(z)} = C_0 + C_1 z + \cdots + C_{q-p-d}z^{q-p-d} + \frac{B_1}{1-z} + \cdots + \frac{B_d}{(1-z)^d} + \sum_{k=1}^{p}\frac{A_k}{1-p_k z}. \quad (3)$$

If p_k is complex then A_k is complex as well and the conjugate fraction $\overline{A}_k/\left(1-\overline{p}_k z\right)$ also appears on the right hand side. The two terms can be combined into the real fraction

$$\frac{\left(A_k + \overline{A}_k\right) - \left(A_k\overline{p}_k + \overline{A}_k p_k\right)z}{1 - \left(p_k + \overline{p}_k\right)z + p_k\overline{p}_k z^2}.$$

After joining complex conjugate fractions, we can express (3) as

$$\frac{\theta(z)}{\phi(z)} = C_0 + C_1 z + \cdots + C_{q-p-d}z^{q-p-d} + \sum_{k=1}^{d}\frac{B_k}{(1-z)^k} + \sum_{k=1}^{m_1}\frac{A_k}{1-p_k z} + \sum_{k=1}^{m_2}\frac{D_k + E_k z}{1 + F_k z + G_k z^2},$$

where the coefficients C_k, B_k, A_k, D_k, E_k, F_k, G_k and p_k are all real.

2.1 Polynomial Methods

It is shown in Gómez and Breitung (1999) that a partial fraction expansion of model (1) leads to the BN decomposition in all cases usually considered in the literature and that, therefore, we can take this expansion as the basis to define the BN decomposition for any ARIMA model. To further justify this approach, consider that in the usual BN decomposition, $y_t = p_t + c_t$, models for the components are obtained that are driven by the same innovations of the series. Thus, if the model for the series is $\phi(B)y_t = \theta(B)a_t$, the models for the components are of the form $\phi_p(B)p_t = \theta_p a_t$ and $\phi_c(B)c_t = \theta_c a_t$. But this implies the decomposition

$$\frac{\theta(z)}{\phi(z)} = \frac{\theta_p(z)}{\phi_p(z)} + \frac{\theta_c(z)}{\phi_c(z)},$$

and, since the denominator polynomials on the right hand side have no roots in common, the previous decomposition coincides with the partial fraction decomposition, that is unique.

Assuming then that the parallel decomposition of the ARIMA model (1) is the basis of the BN decomposition, suppose in (1) that p and P are the degrees of the autoregressive polynomials, $\phi(B)$ and $\Phi(B)$, and q and Q are those of the moving average polynomials, $\theta(B)$ and $\Theta(B)$. Then, letting $\phi^*(B) = \phi(B)\Phi(B^n)$, $\Delta(B) = \nabla^d \nabla_n^D$ and $\theta^*(B) = \theta(B)\Theta(B^n)$, supposing for simplicity that there is no mean in (1) and using Lemma 1, the partial fraction expansion corresponding to model (1) is

$$\frac{\theta^*(z)}{\phi^*(z)\Delta(z)} = \gamma(z) + \frac{\alpha_p(z)}{(1-z)^{d+D}} + \frac{\alpha_s(z)}{S(z)} + \frac{\alpha_c(z)}{\phi^*(z)}, \tag{4}$$

where $S(z) = 1 + z + \cdots + z^{n-1}$ and we have used in (4) the fact that $\nabla_n = (1-B)S(B)$. Here, we have grouped for simplicity several terms in the expansion so that we are only left with the components in (4). For example,

$$\sum_{k=1}^{d+D} \frac{B_k}{(1-z)^k} = \frac{\alpha_p(z)}{(1-z)^{d+D}},$$

etc. Note that the third term on the right of (4) exists only if $D > 0$. The degrees of the $\gamma(z)$, $\alpha_p(z)$, $\alpha_s(z)$ and $\alpha_c(z)$ polynomials in (4) are, respectively, $\max\{0, q^* - p^* - d^*\}$, $d^* - 1$, $n - 2$ and $p^* - 1$, where $p^*{=}p + P$, $q^* = q + Q$ and $d^* = d + D$.

Based on the previous decomposition, we can define several components that are driven by the same innovations, $\{a_t\}$. The assignment of the terms in (4) to the different components depends on the roots of the autoregressive polynomials in (1). For example, the factor $(1-z)^{d^*}$, containing the root one, should be assigned to the trend component, p_t, since it corresponds to an infinite peak in the pseudospectrum of the series at the zero frequency. Since all the roots of the polynomial $S(z)$ correspond to infinite peaks in the pseudospectrum at the seasonal frequencies, the factor $S(z)$ should be assigned to the seasonal component, s_t.

The situation is not so clear–cut, however, as regards the roots of the autoregressive polynomial, $\phi(z)\Phi(z^n)$, and in this case the assignment is more subjective. We will consider for simplicity in the rest of the article only a third component, that will be referred to as "stationary component", c_t. All the roots of $\phi(z)\Phi(z^n)$ will be assigned to this stationary component. Therefore, this component may include cyclical and stationary trend and seasonal components.

According to the aforementioned considerations, the SSOE components model

$$y_t = p_t + s_t + c_t \tag{5}$$

can be defined, where p_t is the trend, s_t is the seasonal and c_t is the stationary component. The models for these components are given by

$$\nabla^{d^*} p_t = \alpha_p(B)a_t, \quad S(B)s_t = \alpha_s(B)a_t, \quad \phi^*(B)c_t = \eta(B)a_t, \tag{6}$$

where $\eta(z) = \gamma(z)\phi^*(z) + \alpha_c(z)$.

Instead of expressing model (1) using the backshift operator, where the time runs backwards, it is possible to use the forward operator, $Fy_t = y_{t+1}$, and let the time run forwards. To this end, let $m = \max\{q^*, p^* + d^*\}$ and $r = \max\{0, q^* - p^* - d^*\}$, where q^*, p^* and d^* are, as defined earlier, the degrees of the polynomials $\theta^*(z)$ and $\phi^*(z)\Delta(z)$ in (4). Then, using again Lemma 1, but with the z^{-1} instead of the z variable, it is obtained that

$$\frac{z^{-m}\theta^*(z)}{z^{-m}\phi^*(z)\Delta(z)} = 1 + \frac{\delta(z^{-1})}{z^{-r}} + \frac{\beta_p(z^{-1})}{(z^{-1}-1)^{d^*}} + \frac{\beta_s(z^{-1})}{S(z^{-1})} + \frac{\beta_c(z^{-1})}{\overline{\phi}^*(z^{-1})}, \tag{7}$$

where $\overline{\phi}^*(z^{-1}) = z^{-p^*}\phi^*(z)$ and the degrees of the polynomials $\delta(z^{-1})$, $\beta_p(z^{-1})$, $\beta_s(z^{-1})$ and $\beta_c(z^{-1})$ are, respectively, $\max\{0, r-1\}$, d^*-1, $n-2$ and p^*-1. Transforming each of the terms of the right hand side of (7) back to the z variable yields

$$\frac{\theta^*(z)}{\phi^*(z)\Delta(z)} = 1 + z\delta(z) + \frac{z\beta_p(z)}{(1-z)^{d^*}} + \frac{z\beta_s(z)}{S(z)} + \frac{z\beta_c(z)}{\phi^*(z)}. \tag{8}$$

This decomposition is an innovations form of the ARIMA model because if we multiply both terms of (8) by the innovation, a_t, we get the equality

$$\begin{aligned} y_t &= a_t + y_{t|t-1} \\ &= a_t + p_{t|t-1} + s_{t|t-1} + c_{t|t-1}, \end{aligned}$$

where, given a random variable x_t, $x_{t|t-1}$ denotes the orthogonal projection of x_t onto $\{y_s : s = t-1, t-2, \ldots\}$. If the series is nonstationary, the orthogonal projection is done onto the finite past of the series plus the initial conditions. The relationship among the components p_t, s_t and c_t and their predictors, $p_{t|t-1}$, $s_{t|t-1}$ and $c_{t|t-1}$, can be obtained by computing the decomposition of each component in the forward operator. For example, if we take the model followed by the trend component given in (6), $\nabla^{d^*} p_t = \alpha_p(B)a_t$, we can write

$$\frac{z^{-d^*}\alpha_p(z)}{z^{-d^*}(1-z)^{d^*}} = k_p + \frac{\beta_p(z^{-1})}{(z^{-1}-1)^{d^*}},$$

where k_p is a constant. Then, returning to the backward operator,

$$\frac{\alpha_p(z)}{(1-z)^{d^*}} = k_p + \frac{z\beta_p(z)}{(1-z)^{d^*}} \tag{9}$$

and multiplying both terms of (9) by the innovation, a_t, yields

$$p_t = k_p a_t + p_{t|t-1}. \tag{10}$$

Therefore, $p_{t|t-1}$ follows the model $\nabla^{d^*} p_{t|t-1} = \beta_p(B) a_{t-1}$. In a similar way, we can prove that there exist constants, k_γ, k_s and k_c such that

$$\gamma(z) = k_\gamma + z\delta(z), \qquad \frac{\alpha_s(z)}{S(z)} = k_s + \frac{z\beta_s(z)}{S(z)}, \qquad \frac{\alpha_c(z)}{\phi^*(z)} = k_c + \frac{z\beta_c(z)}{\phi^*(z)},$$
$$s_t = k_s a_t + s_{t|t-1}, \qquad c_t = (k_\gamma + k_c)\, a_t + c_{t|t-1},$$

and $s_{t|t-1}$ and $c_{t|t-1}$ follow the models

$$S(B) s_{t|t-1} = \beta_s(B) a_{t-1}, \qquad \phi^*(B) c_{t|t-1} = [\phi^*(B)\delta(B) + \beta_c(B)]\, a_{t-1}.$$

Note that the previous relations imply the equality

$$1 = k_\gamma + k_p + k_s + k_c. \tag{11}$$

An example will help clarify matters. Suppose the ARIMA model

$$\nabla_4 y_t = \left(1 - \frac{1}{2}B^5\right) a_t. \tag{12}$$

Then, the BN decomposition is given by the partial fraction decomposition of the model, i.e.

$$\frac{1 - \frac{1}{2}z^5}{1 - z^4} = \frac{1}{2}z + \frac{1}{8}\frac{1}{1-z} + \frac{3}{8}\frac{1}{1+z} + \frac{1}{2}\frac{1 - \frac{1}{2}z}{1 + z^2}.$$

Thus, defining

$$p_t = \frac{1}{1-B}\frac{1}{8}a_t, \qquad s_{1,t} = \frac{1}{1+B}\frac{3}{8}a_t, \qquad s_{2,t} = \frac{1 - \frac{1}{2}B}{1+B^2}\frac{1}{2}a_t, \qquad c_t = \frac{1}{2}a_{t-1}, \tag{13}$$

and $s_t = s_{1,t} + s_{2,t}$, the BN decomposition, $y_t = p_t + s_t + c_t$, is obtained. The innovations form is given by the partial fraction decomposition of the model using the forward operator, i.e.

$$\begin{aligned}\frac{z^{-5} - \frac{1}{2}}{z^{-1}(z^{-4} - 1)} &= 1 + \frac{1}{2}\frac{1}{z^{-1}} + \frac{1}{8}\frac{1}{z^{-1} - 1} - \frac{3}{8}\frac{1}{z^{-1} + 1} - \frac{1}{4}\frac{z^{-1} + 2}{z^{-2} + 1} \\ &= 1 + \frac{1}{2}z + \frac{1}{8}\frac{z}{1-z} - \frac{3}{8}\frac{z}{1+z} - \frac{1}{4}\frac{z + 2z^2}{1 + z^2}.\end{aligned}$$

It follows from this that the innovations form is $y_t = a_t + p_{t|t-1} + s_{1,t|t-1} + s_{2,t|t-1} + c_{t|t-1}$, where

$$p_{t|t-1} = \frac{1}{1-B}\frac{1}{8}a_{t-1}, \qquad s_{1,t|t-1} = -\frac{1}{1+B}\frac{3}{8}a_{t-1},$$
$$s_{2,t|t-1} = -\frac{1+2B}{1+B^2}\frac{1}{4}a_{t-1}, \qquad c_{t|t-1} = \frac{1}{2}a_{t-1}.$$

In addition, the following relations hold

$$p_t = \frac{1}{8}a_t + p_{t|t-1}, \quad s_{1,t} = \frac{3}{8}a_t + s_{1,t|t-1}, \quad s_{2,t} = \frac{1}{2}a_t + s_{2,t|t-1}, \quad c_t = c_{t|t-1}.$$

2.2 State Space Methods

There are many ways to put an ARIMA model into state space form. We will use in this article the one proposed by Akaike (1974). If $\{y_t\}$ follows the ARIMA model $\phi(B)y_t = \theta(B)a_t$, where $\phi(z) = 1+\phi_1 z+\cdots+\phi_p z^p$ and $\theta(z) = \theta_0+\theta_1 z+\cdots+\theta_q z^q$, let $r = \max(p, q+1)$, $\psi(z) = \phi^{-1}(z)\theta(z) = \sum_{j=0}^{\infty}\psi_j z^j$ and $x_{t,1} = y_t$, $x_{t,i} = y_{t+i-1} - \sum_{j=0}^{i-2}\psi_j a_{t+i-1-j}$, $2 \le i \le r$. Then, the following state space representation holds

$$\begin{aligned} x_t &= Fx_{t-1} + K_f a_t \\ y_t &= Hx_t, \end{aligned} \tag{14}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\phi_r & -\phi_{r-1} & -\phi_{r-2} & \cdots & -\phi_1 \end{bmatrix}, \qquad K_f = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{r-2} \\ \psi_{r-1} \end{bmatrix}, \tag{15}$$

$\phi_i = 0$ if $i > p$, $x_t = [x_{t,1}, \ldots, x_{t,r}]'$ and $H = [1, 0, \ldots, 0]$. Note that we are assuming that θ_0 can be different from one, something that happens with the models for the components in the BN decomposition. The representation (14) and (15) is not minimal if $q > p$, but has the advantage that the first element of the state vector is y_t (the other elements of the state vector are the one to $r-1$ periods ahead forecasts of y_t). This is particularly useful if y_t is not observed, so that this representation is adequate to put the BN decomposition into state space form. To see this, suppose that the BN decomposition is $y_t = p_t + s_t + c_t$, where $\{y_t\}$ follows the model (1) and the components follow the models given by (6).

Then, we can set up for each component a state space representation of the form (14) and (15) so that, with an obvious notation, we get the following representation for $\{y_t\}$

$$\begin{aligned}
\begin{bmatrix} x_{p,t} \\ x_{s,t} \\ x_{c,t} \end{bmatrix} &= \begin{bmatrix} F_p & 0 & 0 \\ 0 & F_s & 0 \\ 0 & 0 & F_c \end{bmatrix} \begin{bmatrix} x_{p,t-1} \\ x_{s,t-1} \\ x_{c,t-1} \end{bmatrix} + \begin{bmatrix} K_{f,p} \\ K_{f,s} \\ K_{f,c} \end{bmatrix} a_t \\
y_t &= \begin{bmatrix} H_p & H_s & H_c \end{bmatrix} \begin{bmatrix} x_{p,t} \\ x_{s,t} \\ x_{c,t} \end{bmatrix},
\end{aligned} \tag{16}$$

where $p_t = H_p x_{p,t}$, $s_t = H_s x_{s,t}$ and $c_t = H_c x_{c,t}$. Letting $x_t = [x'_{p,t}, x'_{s,t}, x'_{c,t}]'$, $F = \mathrm{diag}(F_p, F_s, F_c)$, $K_f = [K'_{f,p}, K'_{f,s}, K'_{f,c}]'$ and $H = [H_p,\ H_s,\ H_c]$, we can assume that the state space representation of $\{y_t\}$ is given by (14).

To obtain the innovations state space model corresponding to (14), where $x_t = [x'_{p,t}, x'_{s,t}, x'_{c,t}]'$, $F = \mathrm{diag}(F_p, F_s, F_c)$, $K_f = [K'_{f,p}, K'_{f,s}, K'_{f,c}]'$ and $H = [H_p,\ H_s,\ H_c]$ satisfy (16), consider first that in terms of the matrices in (14) the transfer function, $\psi(z)$, of model (1) can be expressed as

$$\begin{aligned}
\psi(z) &= H\,(I - Fz)^{-1}\,K_f \\
&= 1 + zH\,(I - Fz)^{-1}\,FK_f
\end{aligned} \tag{17}$$

and thus the following relation holds

$$HK_f = 1 \tag{18}$$

This relation is the state space equivalent to the polynomial relation (11). We also get from (17) that

$$y_t = a_t + H\,(I - FB)^{-1}\,FK_f a_{t-1},$$

where B is the backshift operator. Then, if we define

$$K = FK_f, \tag{19}$$

and

$$x_{t+1|t} = (I - FB)^{-1}\,FK_f a_t,$$

we obtain the following state space representation

$$\begin{aligned}
x_{t+1|t} &= Fx_{t|t-1} + Ka_t \\
y_t &= Hx_{t|t-1} + a_t.
\end{aligned} \tag{20}$$

Note that $x_{t|t-1}$ is the projection of x_t onto $\{y_{t-1}, y_{t-2}, \ldots, y_1, x_1\}$ because

$$\begin{aligned} x_t &= (I - FB)^{-1} K_f a_t = \left[K_f + z(I - FB)^{-1} F K_f\right] a_t \\ &= x_{t|t-1} + K_f a_t. \end{aligned} \tag{21}$$

In fact, (21) is the measurement update formula corresponding to (20). Therefore, $y_{t|t-1} = Hx_{t|t-1}$ and the equations (20) constitute an innovations state space representation for $y_t = p_t + s_t + c_t$ such that $y_t = p_{t|t-1} + s_{t|t-1} + c_{t|t-1} + a_t$.

If the ARIMA model followed by y_t is invertible, so is its transfer function. In this case, by the matrix inversion lemma applied to (17), it is obtained that

$$\psi(z)^{-1} = 1 - zH(I - F_p z)^{-1} K,$$

where $F_p = F - KH$ has all its eigenvalues inside the unit circle. In fact, it can be shown that the eigenvalues of F_p coincide with the inverses of the roots of the moving average polynomial of the model, $\theta(z)$, see, for example, Hannan and Deistler (1988, pp. 97–98).

As an example, we will use again model (12). According to the models (13), the state space form (16) is

$$\begin{bmatrix} p_t \\ s_{1,t} \\ s_{2,t} \\ s_{2,t+1|t} \\ c_t \\ c_{t+1|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ s_{1,t-1} \\ s_{2,t-1} \\ s_{2,t|t-1} \\ c_{t-1} \\ c_{t|t-1} \end{bmatrix} + \begin{bmatrix} 1/8 \\ 3/8 \\ 1/2 \\ -1/4 \\ 0 \\ 1/2 \end{bmatrix} a_t \tag{22}$$

$$y_t = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ s_{1,t} \\ s_{2,t} \\ s_{2,t+1|t} \\ c_t \\ c_{t+1|t} \end{bmatrix} \tag{23}$$

Using (19), the innovations state space form is

$$\begin{bmatrix} p_{t+1|t} \\ s_{1,t+1|t} \\ s_{2,t+1|t} \\ s_{2,t+2|t} \\ c_{t+1|t} \\ c_{t+2|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{t|t-1} \\ s_{1,t|t-1} \\ s_{2,t|t-1} \\ s_{2,t+1|t-1} \\ c_{t|t-1} \\ c_{t+1|t-1} \end{bmatrix} + \begin{bmatrix} 1/8 \\ -3/8 \\ -1/4 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix} a_t \tag{24}$$

$$y_t = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{t|t-1} \\ s_{1,t|t-1} \\ s_{2,t|t-1} \\ s_{2,t+1|t-1} \\ c_{t|t-1} \\ c_{t+1|t-1} \end{bmatrix} + a_t. \tag{25}$$

Note that the relation $HK_f = 1$ holds and that the matrix $F_p = F - KH$ has all its eigenvalues inside the unit circle. Note also that the last row in the transition equation is zero and that, therefore, the last state can be eliminated from the state space form. In this way, we would obtain a minimal innovations state space form.

Suppose we are given an innovations state space representation (20), minimal or not, where $x_{t|t-1} = [x'_{p,t|t-1}, x'_{s,t|t-1}, x'_{c,t|t-1}]'$, $F = \text{diag}(F_p,\ F_s,\ F_c)$, $K = [K'_p, K'_s, K'_c]'$ and $H = [H_p,\ H_s,\ H_c]$, and we want to obtain the BN decomposition, $y_t = p_t + s_t + c_t$, in state space form. We assume that F_p and F_s are nonsingular and F_c may be singular or empty. Of course, if F_c is empty, so are $x_{c,t|t-1}$, K_c and H_c. Note that F_p and F_s are the matrices containing the unit and the seasonal roots, respectively, and that if F_c is singular or empty, then $\gamma(z)$ in (4) is nonzero. To obtain the BN decomposition we distinguish two cases, depending on whether F_c is singular or empty or nonsingular. If F_c is nonsingular, then we solve for K_f in

$$FK_f = K$$

to get (14), where $p_t = H_p x_{p,t}$, $s_t = H_s x_{s,t}$ and $c_t = H_c x_{c,t}$. If F_c is singular, then defining

$$c_t = c_{t|t-1} + k_c a_t,$$

where $c_{t|t-1} = H_c x_{c,t|t-1}$ and k_c is a constant, we can write

$$\begin{matrix} c_t \\ x_{c,t+1|t} \end{matrix} = \begin{matrix} 0 & H_c \\ 0 & F_c \end{matrix} \quad \begin{matrix} c_{t-1} \\ x_{c,t|t-1} \end{matrix} + \begin{matrix} k_c \\ K_c \end{matrix} \ a_t.$$

If F_c is empty, then the previous expressions collapse to $c_t = k_c a_t$. In the following, we will only consider the case in which F_c is singular, leaving to the reader the necessary changes if F_c is empty. Thus, if we further define $x^a_{c,t} = [c_t, x'_{c,t+1|t}]'$, $H^a_c = [1, 0]$, $K^a_c = [0, K'_c]'$, $F^a_c = [0, C_c]$, where $C_c = [H'_c, F'_c]'$ and H^a_c, K^a_c and F^a_c are conformal with $x^a_{c,t}$, we can write

$$\begin{bmatrix} x_{p,t+1|t} \\ x_{s,t+1|t} \\ x^a_{c,t+1|t} \end{bmatrix} = \begin{bmatrix} F_p & 0 & 0 \\ 0 & F_s & 0 \\ 0 & 0 & F^a_c \end{bmatrix} \begin{bmatrix} x_{p,t|t-1} \\ x_{s,t|t-1} \\ x^a_{c,t|t-1} \end{bmatrix} + \begin{bmatrix} K_p \\ K_s \\ K^a_c \end{bmatrix} a_t$$

$$y_t = \begin{bmatrix} H_p & H_s & H^a_c \end{bmatrix} \begin{bmatrix} x_{p,t|t-1} \\ x_{s,t|t-1} \\ x^a_{c,t|t-1} \end{bmatrix} + a_t.$$

Solving for K^a_f in

$$F^a K^a_f = K^a, \qquad H^a K^a_f = 1,$$

where $F^a = \text{diag}(F_p,\, F_s,\, F^a_c)$, $K^a = [K'_p, K'_s,\, K^{a\prime}_c]'$, $H^a = [H_p,\, H_s,\, H^a_c]$, $K^a_f = [K'_{f,p}, K'_{f,s}, K^{a\prime}_{f,c}]'$ and $K^a_{f,c} = [k_c, K'_{f,c}]'$, we get (14), where $p_t = H_p x_{p,t}$, $s_t = H_s x_{s,t}$, $x_{c,t} = x^a_{c,t}$ and $c_t = H^a_c x_{c,t}$. Note that $k_c = 1 - H_p K_{f,p} - H_s K_{f,s}$ and that (14) is not minimal in this case.

As an example, the reader can verify that if we start with (24) and (25), we eliminate the last state in those equations, and we follow the previous procedure, then we get (22) and (23).

2.3 Connection With Exponential Smoothing

There has been lately some interest in using generalized exponential smoothing models for forecasting (see De Livera et al., 2011). These models are SSOE models that once they are put into state space form they become innovations model of the type we have considered in earlier sections. The question then arises as to whether these models have any connection with the models given by a parallel decomposition of an ARIMA model. It turns out that many of the basic exponential smoothing models coincide with those corresponding to the BN decomposition and in those cases, mostly seasonal, where they do not coincide the exponential smoothing models have been shown to have some kind of problem that is solved if the models given by the parallel decomposition are used instead.

To see this, suppose first Holt's linear model, $y_t = p_{t-1} + b_{t-1} + a_t$, where

$$
\begin{aligned}
p_t &= p_{t-1} + b_{t-1} + k_1 a_t & (26)\\
b_t &= b_{t-1} + k_2 a_t, & (27)
\end{aligned}
$$

and k_1 and k_2 are constants. If we substitute (26) into the expression for y_t, it is obtained that $y_t = p_t + (1 - k_1)\, a_t$. In addition, it follows from (26) and (27) that

$$\nabla^2 p_t = k_2 a_{t-1} + k_1 \nabla a_t.$$

Therefore, we can write

$$y_t = \left[\frac{\alpha_p(B)}{(1-B)^2} + k_c \right] a_t, \tag{28}$$

where $\alpha_p(z) = k_2 z + k_1(1-z)$ and $k_c = 1 - k_1$. Since the partial fraction expansion of the polynomial in the backshift operator on the right hand side of (28) is

$$\frac{k_2 z + k_1(1-z)}{(1-z)^2} + k_c = \frac{k_1 - k_2}{1-z} + \frac{k_2}{(1-z)^2} + k_c,$$

if we define $c_t = k_c a_t$, then $y_t = p_t + c_t$, $p_{t|t-1} = p_{t-1} + b_{t-1}$, $c_{t|t-1} = 0$ and $y_t = p_{t|t-1} + a_t$. Thus, it is seen that Holt's linear model is the innovations form corresponding to the BN decomposition of an ARIMA model,

$$\nabla^2 y_t = (1 + \theta_1 B + \theta_2 B^2) a_t, \tag{29}$$

where $\theta_1 = k_1 + k_2 - 2$ and $\theta_2 = 1 - k_1$. Note that, since k_1 and k_2 can univocally be solved in terms of θ_1 and θ_2 in the previous expressions, every ARIMA model (29) can be put in the form of a Holt's model, $y_t = p_{t|t-1} + a_t$, where $p_{t|t-1} = p_{t-1} + b_{t-1}$ and p_t and b_t are given by (26) and (27).

Suppose now Holt–Winters' model, $y_t = p_{t-1} + b_{t-1} + s_{t-n} + a_t$, where

$$
\begin{aligned}
p_t &= p_{t-1} + b_{t-1} + k_1 a_t & (30)\\
b_t &= b_{t-1} + k_2 a_t, & (31)\\
s_t &= s_{t-n} + k_3 a_t, & (32)
\end{aligned}
$$

and k_1, k_2 and k_3 are constants. There are apparently three unit roots in the model. However, a closer look will reveal that there are in fact only two unit roots. To see this, substitute (30) and (32) into the expression for y_t to give $y_t = p_t + s_t + (1 - k_1 - k_3)\, a_t$. In addition, it follows from (30), (31) and (32) that

$$\nabla^2 p_t = k_2 a_{t-1} + k_1 \nabla a_t, \qquad \nabla_n s_t = k_3 a_t.$$

Then, we can write

$$y_t = \left[\frac{\alpha_p(B)}{(1-B)^2} + \frac{\alpha_s(B)}{(1-B)S(B)} + k_c \right] a_t, \tag{33}$$

where $S(z) = 1 + z + \cdots + z^{n-1}$, $\alpha_p(z) = k_2 z + k_1(1-z)$, $\alpha_s(z) = k_3$ and $k_c = 1 - k_1 - k_3$. The partial fraction expansion of the polynomial in the backshift operator on the right hand side of (33) is

$$\frac{k_2 z + k_1(1-z)}{(1-z)^2} + \frac{k_3}{(1-z)S(z)} + k_c = \frac{k_1 - k_2}{1-z} + \frac{k_2}{(1-z)^2} + \frac{k_3/n}{1-z} + \frac{\beta(z)}{S(z)} + k_c,$$

where $\beta(z) = [(n-1) + (n-2)z + \cdots + 2z^{n-3} + z^{n-2}]\, k_3/n$. Thus, if we define $c_t = k_c a_t$, then $y_t = p_t + s_t + c_t$, $p_{t|t-1} = p_{t-1} + b_{t-1}$, $s_{t|t-1} = s_{t-n}$, $c_{t|t-1} = 0$ and $y_t = p_{t|t-1} + s_{t|t-1} + a_t$. Therefore, Holt–Winters' model is the innovations form corresponding to the BN decomposition of an ARIMA model of the form

$$\nabla^2 S(B) y_t = \theta(B) a_t, \tag{34}$$

where $\theta(z)$ is a polynomial of degree $n+1$. However, the components are not well defined because the seasonal component can be further decomposed as

$$s_t = \left[\frac{k_3/n}{1-B} + \frac{\beta(B)}{S(B)} \right] a_t,$$

and we see that the first subcomponent should be assigned to the trend because the denominator has a unit root. To remedy this problem, the seasonal component should be defined as the second subcomponent only, so that Holt-Winters' method should be modified to a model of the form $y_t = p_{t|t-1} + s_{t|t-1} + a_t$, where p_t and b_t are given by (30), (31),

$$s_t = -\sum_{i=1}^{n-1} s_{t-i} + \beta(B) a_t,$$

$p_{t|t-1} = p_{t-1} + b_{t-1}$ and $s_{t|t-1} = -\sum_{i=1}^{n-1} s_{t-i}$. The model can be simplified if we assume $\beta(B) = k_3$. Another possibility is to decompose the seasonal component further according to the partial fraction expansion of its model,

$$\frac{\beta(z)}{S(z)} = \sum_{i=1}^{[n/2]} \frac{k_{i,1} + k_{i,2} z}{1 - 2\alpha_i z + z^2} \tag{35}$$

where $[x]$ denotes the greatest integer less than or equal x, $\alpha_i = \cos\omega_i$ and $\omega_i = 2\pi i/n$ is the i-th seasonal frequency. If n is even, $\omega_{n/2} = 2\pi[n/2]/n = \pi$ and the corresponding term in the sum on the right hand side of (35) collapses to $k_{n/2}/(1+z)$. This would lead us to a seasonal component of the form

$$
\begin{aligned}
s_t &= \sum_{i=1}^{[n/2]} s_{i,t} &(36)\\
(1 - 2\alpha_i B + B^2)s_{i,t} &= (k_{i,1} + k_{i,2}B)a_t, &(37)
\end{aligned}
$$

where we can assume $k_{i,1} = k_1$ and $k_{i,2} = k_2$ for parsimony. It is a consequence of the partial fraction decomposition that there is a bijection between ARIMA models of the form (34) and exponential smoothing models, $y_t = p_{t|t-1} + s_{t|t-1} + a_t$, where p_t and s_t are given by (30), (31), (36) and (37). A solution similar to (36) and (37) has been suggested by De Livera et al. (2011), where they propose for each component, s_{it}, the model

$$
\begin{matrix} s_{i,t} \\ s^*_{i,t} \end{matrix} = \begin{matrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{matrix} \quad \begin{matrix} s_{i,t-1} \\ s^*_{i,t-1} \end{matrix} + \begin{matrix} r_1 \\ r_2 \end{matrix} a_t. \qquad (38)
$$

We will see in the next section that both solutions are in fact equivalent. However, in De Livera et al. (2011, p. 1516, p. 1520) the expression $y_t = p_{t-1} + b_{t-1} + s_{t-1} + a_t$ is used. This implies $s_{i,t|t-1} = s_{i,t-1}$ in model (38), something that is incorrect. The correct expression can be obtained using the method described in Section 2.2 and, more specifically, formula (19). Thus,

$$
\begin{matrix} s_{i,t+1|t} \\ s^*_{i,t+1|t} \end{matrix} = \begin{matrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{matrix} \quad \begin{matrix} s_{i,t|t-1} \\ s^*_{i,t|t-1} \end{matrix} + \begin{matrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{matrix} \quad \begin{matrix} r_1 \\ r_2 \end{matrix} a_t.
$$

The expression for $s_{i,t|t-1}$ corresponding to $s_{i,t}$ in (37) can be obtained by expanding model (37) using the forward operator, as described in Section 2.1.

2.4 Specification for Complex Models Using the Beveridge–Nelson Decomposition

Sometimes, there are time series that follow ARIMA models with complex seasonal patterns, like model (2). This is the case with hourly or daily data, that exhibit several seasonal patterns, some of them with even non integer seasonal periods.

Suppose first the ARIMA model

$$\phi(B)\nabla^2 S(B)y_t = \theta(B)a_t, \tag{39}$$

where $S(z) = 1 + z + \cdots + z^{n-1}$ and the polynomials $\phi(z)$ and $\theta(z)$ have all their roots outside the unit circle and have degrees p and q, respectively. Then, its partial fraction expansion is

$$\frac{\theta(z)}{\phi(z)\nabla^2 S(z)} = \gamma(z) + \frac{r_1}{1-z} + \frac{r_2}{(1-z)^2} + \sum_{i=1}^{[n/2]} \frac{r_{i,1} + r_{i,2}z}{1 - 2\alpha_i z + z^2} + \frac{\beta_c(z)}{\phi(z)}, \tag{40}$$

where $\alpha_i = \cos\omega_i$, $\omega_i = 2\pi i/n$ is the i-th seasonal frequency and $\beta_c(z)$ and $\gamma(z)$ have degrees $p-1$ and $\max\{0, q-p-n-1\}$, respectively. Given that the models in (40) are simpler than the original model (39), the question arises as to whether it is better to specify the model starting with (40) rather than using (39). This motivates that we define the components

$$p_t = p_{t-1} + b_{t-1} + k_1 a_t \tag{41}$$

$$b_t = b_{t-1} + k_2 a_t \tag{42}$$

$$s_t = \sum_{i=1}^{[n/2]} s_{i,t} \tag{43}$$

$$(1 - 2\alpha_i B + B^2)s_{i,t} = (r_{i,1} + r_{i,2}B)a_t \tag{44}$$

$$\phi(B)c_t = \theta^*(B)a_t, \tag{45}$$

where $k_1 = r_1 + r_2$, $k_2 = r_2$ and $\theta^*(z) = \gamma(z)\phi(z) + \beta_c(z)$, so that the BN decomposition is

$$y_t = p_t + s_t + c_t. \tag{46}$$

To achieve parsimony, we can set in (44) $r_{i,1} = k_3$ and $r_{i,2} = k_4$.

The specification (46) with the components following the models (41) to (45) has some similarities to the generalized exponential smoothing TBATS model proposed by De Livera et al. (2011). As we saw in Section 2.3, all linear exponential smoothing models should be the innovations form of some BN decomposition. However, in the TBATS models the innovations in (41) to (44) appear replaced with the stationary component,

c_t. In addition, as we saw also in Section 2.3, there is an inconsistency in the definition of the seasonal component in the TBATS models.

We now turn to the state space form of the BN decomposition (46). The model for the trend, p_t, can be put into state space form as follows,

$$x_{p,t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{p,t-1} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} a_t \tag{47}$$

$$p_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{p,t}, \tag{48}$$

where $x_{p,t} = [p_t, b_t]'$. If we define $\bar{x}_{p,t} = P x_{p,t}$, where

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

a change of variables in (47)–(48) leads to the state space form proposed by Akaike (1974),

$$\bar{x}_{p,t} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \bar{x}_{p,t-1} + \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \end{bmatrix} a_t \tag{49}$$

$$p_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}_{p,t}, \tag{50}$$

where $\bar{x}_{p,t} = [p_t, p_{t+1|t}]$, $\bar{k}_1 = r_1 + r_2$ and $\bar{k}_2 = r_1 + 2r_2$. The model (44) followed by the subcomponents, $s_{i,t}$, of the seasonal component, s_t, can be put into Akaike's state space form as usual,

$$x_{si,t} = \begin{bmatrix} 0 & 1 \\ -1 & 2\alpha_i \end{bmatrix} x_{si,t-1} + \begin{bmatrix} k_{i,1} \\ k_{i,2} \end{bmatrix} a_t \tag{51}$$

$$s_{i,t} = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{si,t}, \tag{52}$$

where $x_{si,t} = [s_{i,t}, s_{i,t+1|t}]$, $k_{i,1} = r_{i,1}$ and $k_{i,2} = r_{i,2} + 2\alpha_i r_{i,1}$. If we make the change of variables $\bar{x}_{si,t} = P x_{si,t}$ in (51)–(52), where

$$P = \begin{bmatrix} 1 & 0 \\ -\alpha_i/\sin\omega_i & 1/\sin\omega_i \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & 0 \\ \alpha_i & \sin\omega_i \end{bmatrix},$$

the following state space form is obtained,

$$\bar{x}_{si,t} = \begin{bmatrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{bmatrix} \bar{x}_{si,t-1} + \begin{bmatrix} \bar{k}_{i,1} \\ \bar{k}_{i,2} \end{bmatrix} a_t \tag{53}$$

$$s_{i,t} = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}_{si,t}. \tag{54}$$

where $\bar{x}_{si,t} = [s_{i,t}, s^*_{i,t}]$, $s^*_{i,t}$ is an auxiliary random variable, $\bar{k}_{i,1} = r_{i,1}$ and $\bar{k}_{i,2} = r_{i,2}/\sin\omega_i$ $+\alpha_i r_{i,1}/\sin\omega_i$. Finally, the stationary component, c_t, that follows model (45), can be put into Akaike's state space form as in (14)–(15).

Using the previous models for the components, and with an obvious notation, we get a state space form like (16). As shown earlier in this article, the corresponding innovations state space form is a generalized exponential smoothing model. It can be easily obtained by applying the procedure described in Subsection 2.2 or, more specifically, by using formula (19) to get (20). Note that the model for the trend can be either (47)–(48) or (49)–(50), and for the seasonal subcomponents we can use either model (51)–(52) or (53)–(54). As an example, consider again model (12). The state space form (16) is given by (22)–(23), and the corresponding innovations state space form is (24)–(25).

The case of complex seasonal patterns can be handled by replacing s_t in (43) with s_t $= \sum_{j=1}^{N} s^j_t$, where the seasonal component,

$$s^j_t = \sum_{i=1}^{m_j} s^j_{i,t},$$

has seasonal period n_j and the subcomponents $s^j_{i,t}$ follow the model (44) with $s_{i,t}$, α_i, $r_{i,1}$ and $r_{i,2}$ replaced with $s^j_{i,t}$, $\cos(2\pi i/n_j)$, $r^j_{i,1}$ and $r^j_{i,2}$. Here, m_j is the number of harmonics required for the j-th seasonal component. In the cases considered so far, it was $m_j =$ $[n_j/2]$, but with complex seasonal patterns it can be much smaller. The overall ARIMA model corresponding to this model is of the form

$$\phi(B)\left[\nabla^d \prod_{j=1}^{N}\prod_{i=1}^{m_j}(1 - 2\alpha^j_i B + B^2)y_t - \mu\right] = \theta(B)a_t, \tag{55}$$

where $d = 0, 1, 2$ and $\alpha^j_i = \cos(2\pi i/n_j)$. As described earlier, we can set in the previous model $r^j_{i,1} = r^j_1$ and $r^j_{i,2} = r^j_2$ to achieve parsimony.

2.5 Estimation and Forecasting Using the Beveridge–Nelson Decomposition

Suppose that we have specified a state space model (16) as described in the previous section and that we have obtained its innovations state space form (20) by using formula (19). Once the model is in innovations form (20), we can apply the Kalman filter to evaluate the likelihood. Since the model is nonstationary, the ordinary Kalman filter cannot be applied. We can use the diffuse Kalman filter (DKF) of de Jong (1991) instead, that allows for partially diffuse initial conditions. Given that p_t and s_t are nonstationary, if $\mathrm{Var}(a_t) = \sigma^2$, the initial state vector would be $x_{1|0} = A\delta + c$, where $A = [I, 0]'$, δ is a diffuse vector conformal with $[x'_{p,t}, x'_{s,t}]'$ and c is a zero mean stochastic vector with covariance matrix, V, satisfying

$$V = F_c V F_c' + (F_c K_{f,c})(F_c K_{f,c})'\sigma^2.$$

If we consider the whole initial state, $x_{1|0}$, as fixed, we obtain the so–called *conditional* likelihood. In this case, the DKF has a very simple form. Given the sample, $y = (y_1, \ldots, y_n)'$, and letting $x_{1|0} = \beta$, the DKF is given for $t = 1, \ldots, n$ by the recursions

$$\begin{aligned} (E_t, e_t) &= (0, y_t) - H(-U_t, x_{t|t-1}) && (56)\\ \Sigma_t &= HP_tH' + \sigma^2, \quad K_t = (FP_tH' + K\sigma^2)\Sigma_t^{-1} \\ (-U_{t+1}, x_{t+1|t}) &= F(-U_t, x_{t|t-1}) + K_t(E_t, e_t) && (57)\\ P_{t+1} &= (F - K_tH)P_tF' + (K\sigma^2 - K_t\sigma^2)K', \end{aligned}$$

with initial conditions $(-U_1, x_{1|0}) = (-I, 0)$ and $P_1 = 0$. Thus, for all t, $P_t = 0$, $\Sigma_t = \sigma^2$, and $K_t = K$, so that the previous recursions collapse to (56) and (57) with $K_t = K$. Note that (56) and (57) without the augmented part coincide with the equations (20). Instead of using the additional DKF recursion $(S_{t+1}, s_{t+1}) = (S_t, s_t) + \sigma^{-2}E_t'(E_t, e_t)$, initialized with $(S_1, s_1) = (0, 0)$, to estimate the initial state, $x_{1|0} = \beta$, we can proceed in two steps. In the first step, (56) and (57) are run and the quantities e_t and E_t are stored. In the second step, β is estimated in the regression model $e = E\beta + a$, where $e = (e_1, \ldots, e_n)'$, $E = (E_1', \ldots, E_n')'$, $a = (a_1, \ldots, a_n)'$ and $\mathrm{Var}(a) = \sigma^2 I$. In this way, β can be concentrated out of the conditional likelihood. The conditional log–likelihood is, apart from a constant,

$$\lambda(y) = -\frac{1}{2}\left\{n \ln|\sigma^2| + (e - E\hat{\beta})'(e - E\hat{\beta})/\sigma^2\right\},$$

where $\hat{\beta} = (E'E)^{-1}E'e$. The parameter σ^2 can be concentrated out of the conditional log–likelihood and the σ^2–maximized conditional log–likelihood, denoted by $\lambda(y; \widehat{\sigma}^2)$, is

$$\lambda(y; \widehat{\sigma}^2) = \text{constant} - \frac{1}{2}n \ln\left[(e - E\hat{\beta})'(e - E\hat{\beta})\right],$$

where $\widehat{\sigma}^2 = (e - E\hat{\beta})'(e - E\hat{\beta})/n$.

To estimate model (55), we have to maximize $\lambda(y; \widehat{\sigma}^2)$ with respect to the parameters of the model. The stationary autoregressive parameters are in matrix F and correspond to the coefficients of the polynomial $\phi(z)$. They can be kept in the stationarity region during the estimation process using standard procedures. However, it is more difficult to keep the parameters of the moving average polynomial, $\theta(z)$, within the invertibility region, especially if the model has been specified using the parallel decomposition. As mentioned earlier in Section 2.2, the eigenvalues of the matrix $F_p = F - KH$ coincide with the inverses of the roots of $\theta(z)$. Since the parameters of the moving average part are all in matrix K, one way to enforce that these parameters are in the invertibility region is to obtain first the eigenvalues of F_p, using the stable Schur decomposition for example. If there are some eigenvalues that have modulus greater than one, we invert them to get matrix $\overline{F}_p$, say. Then, we obtain a new vector K as $\overline{K} = (F - \overline{F}_p)H'(HH')^{-1}$ and new moving average parameters from it.

Once all of the parameters in the model (55) have been estimated, we can apply the Kalman filter to obtain the forecasts in the usual way. Assuming the innovations have a Gaussian distribution, we can also obtain confidence intervals for the forecasts.

2.6 Signal Extraction Using the Beveridge–Nelson Decomposition

Suppose the BN decomposition (5) corresponding to model (1). We can estimate the unobserved components by expressing the innovations, a_t, in their models (6) in terms of the series, y_t. In the case of the trend, for example, the estimator is

$$\hat{p}_t = \frac{\alpha_p(B)\phi(B)\Phi(B^n)S(B)}{\theta(B)\Theta(B^n)} y_t = \sum_{i=0}^{\infty} \pi_i \hat{y}_{t-i},$$

where $\hat{y}_t = y_t$ if $t = 1, \ldots, n$ and $\hat{y}_t$ is the backcast of y_t if $t \leq 0$. Note that, because the model is assumed invertible, $\sum_{i=0}^{\infty} |\pi_i| < \infty$. For the example (12), the trend estimator is

$$p_t = \frac{1 + B + B^2 + B^3}{1 - (1/2)B^5} \frac{1}{8} y_t.$$

To obtain the estimators given by the BN decomposition, one can use the state space form (16). Once the model is in state space form, the DKF and its corresponding smoother give the component estimators and their mean squared errors.

It is to be noticed that if the semi–infinite sample, $\{\ldots, y_1, y_2, \ldots, y_N\}$, is available, the filters that result from the BN decomposition to estimate the unobserved components are asymmetrical. In this case, they are appropriate only to obtain the concurrent estimators of these components. If estimators based on the doubly–infinite sample, $\{\ldots, y_1, y_2, \ldots, y_N, \ldots\}$, are required, then one should use MSOE models, for which the Wiener–Kolmogorov filters to estimate the components are symmetrical. To see the connection between MSOE and SSOE models, suppose the MSOE model

$$x_{t+1} = Fx_t + Gu_t \tag{58}$$

$$y_t = Hx_t + v_t, \qquad t = 1, \ldots, n, \tag{59}$$

where

$$E\left\{ \begin{matrix} u_t \\ v_t \end{matrix} \; [u_s', v_s'] \right\} = \begin{matrix} Q & S \\ S' & R \end{matrix} \; \delta_{ts},$$

$E(u_t) = 0$, $E(v_t) = 0$, the initial state vector, x_1, is orthogonal to u_t and v_t for all t, $E(x_1) = 0$ and $\text{Var}(x_1) = \Omega$. For example, consider the simple MSOE model

$$p_t = p_{t-1} + d_t$$

$$s_t = -s_{t-1} + e_t,$$

and $y_t = p_t + s_t + c_t$, where p_t is the trend, s_t is the seasonal component and the sequences $\{d_t\}$, $\{e_t\}$ and $\{c_t\}$ have zero mean and are mutually and serially uncorrelated with $\text{Var}(d_t) = \sigma_d^2$, $\text{Var}(e_t) = \sigma_e^2$ and $\text{Var}(c_t) = \sigma_c^2$. Then, $x_t = [p_t, s_t]'$, $u_t = [d_t, e_t]'$, $v_t = c_t$ and

$$F = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}, \quad G = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad Q = \begin{matrix} \sigma_d^2 & 0 \\ 0 & \sigma_e^2 \end{matrix} \quad H = [1, 1], \quad R = \sigma_c^2, \quad S = 0.$$

Assuming that the conditions for the existence of a unique solution, P, of the discrete algebraic Riccati equation corresponding to (58)–(59),

$$P = FPF' + GQG' - (FPH' + GS)(R + HPH')^{-1}(FPH' + GS)', \tag{60}$$

are satisfied, the steady state form of the Kalman filter is given by the equations (20) with $\text{Var}(a_t) = \Sigma$, where $x_{t|t-1}$ is the estimator of x_t based on the semi–infinite sample $\{\ldots, y_1, y_2, \ldots, y_{t-1}\}$, and

$$\Sigma = R + HPH', \quad K = (FPH' + GS)\Sigma^{-1}, \quad P = FPF' + GQG' - K\Sigma K'. \tag{61}$$

Equations (20) are the innovations form corresponding to the reduced form ARIMA model of (58)–(59). The formula for the filtered estimators, also called measurement update, corresponding to the steady state is

$$x_{t|t} = x_{t|t-1} + K_f a_t,$$

where $K_f = PH'\Sigma^{-1}$. It is immediately seen from the equations of the time and measurement updates corresponding to the steady state of the Kalman filter that if $S = 0$, then $FK_f = K$, $x_{t+1|t} = Fx_{t|t}$ and

$$\begin{aligned} x_{t|t} &= Fx_{t-1|t-1} + K_f a_t \\ y_t &= Hx_{t|t} + (1 - HK_f)\, a_t. \end{aligned} \tag{62}$$

If we define $c_{t|t} = (1 - HK_f)\, a_t$ and redefine equations (62) so that the state vector $x_{t|t}$ is enlarged to $[x'_{t|t}, c_{t|t}]'$, then these redefined equations constitute in fact the state space form of the BN decomposition of the reduced form ARIMA model of (58)–(59). This can be verified as in Section 2.2, when passing from an innovations state space form to the state space form of the BN decomposition. Note that we have obtained (62) assuming $S = 0$, that is, that the signal, $Hx_t = p_t + s_t$, is uncorrelated with the noise, v_t. Note also that in (62) the BN components are *defined* as the filtered estimators of the components in (58)–(59). This implies that the filters for the concurrent estimators based on the semi–infinite sample given by the BN decomposition corresponding to (62) coincide with those given by (58)–(59).

If we want symmetric filters to estimate the components in (58)–(59) using smoothing and the doubly–infinite sample, $\{\ldots, y_1, \ldots, y_N, \ldots\}$, in addition to assuming $S = 0$, we should further assume that p_t, s_t and c_t are orthogonal.

3 Application

In this section, we use the specification based on the BN decomposition described earlier on Section 2.4 with three real series that have different seasonal patterns. The series are the monthly airline series of Box and Jenkins (1976), the quarterly U.S. real GDP series of Oh et al. (2008) and the weekly U.S. Gasoline Data of De Livera et al. (2011). The three series can be seen in Figure 1.

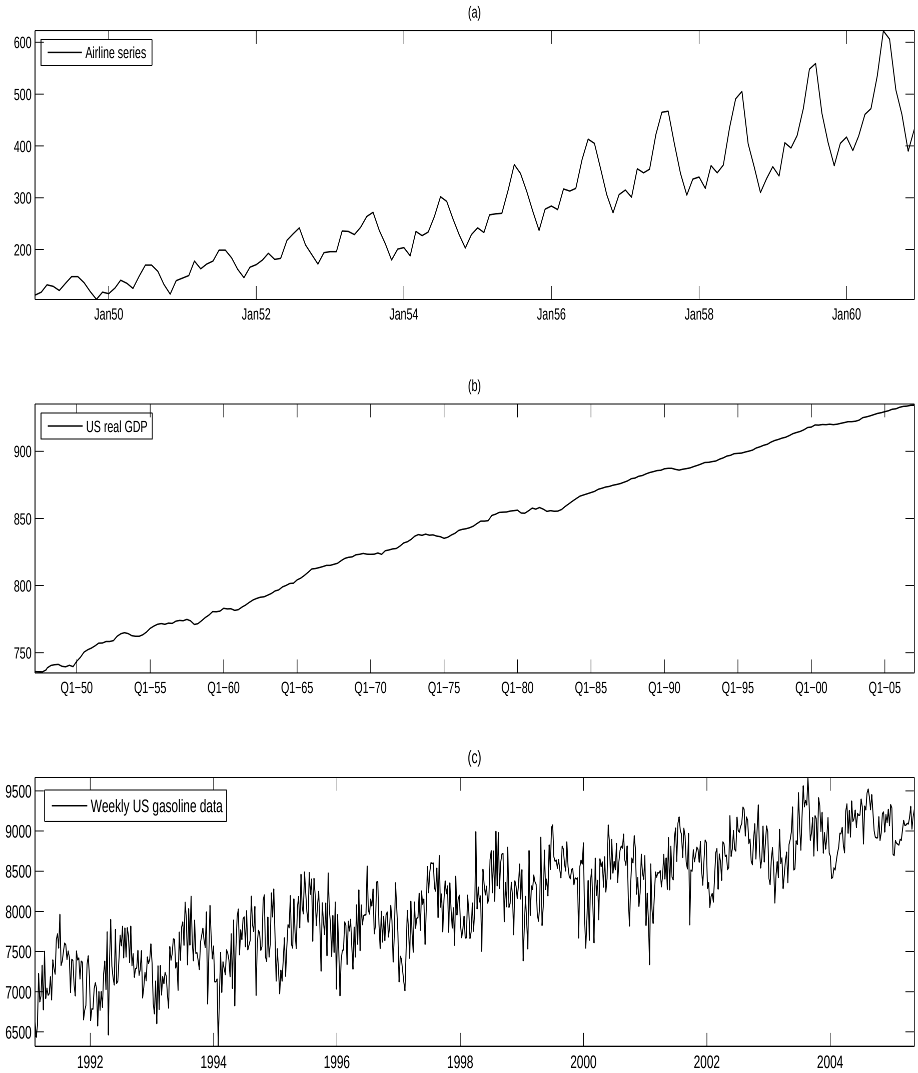

Figure 1: Series With Different Seasonal Patterns. (a) Monthly airline series of Box and Jenkins (1976). (b) Quarterly U.S. real GDP series of Oh et al. (2008). (c) Weekly U.S. Gasoline Data of De Livera et al. (2011).

Each series was split into two parts: an estimation sample period and a sample to compare forecasts with actual values. The estimation sample was used to obtain the conditional likelihood estimates of the model parameters.

The specification for the logged monthly airline series consists of equations (41)–(42)

for the trend, and equation (43), where $n = 12$ and $s_{i,t}$ follows (53)–(54) with $\bar{k}_{i,1} = \bar{k}_1$ and $\bar{k}_{i,2} = \bar{k}_2$, for the seasonal component. The stationary component, c_t, is white noise. The last 48 observations were reserved to assess forecasting performance.

The logged quarterly U.S. real GDP series is assumed to follow one of the models in Oh et al. (2008), namely ARIMA$(0, 2, 2)$. Thus, the specification consists of equations (41)–(42) for the trend, and a stationary component following white noise. The last 12 observations were hold out to compare forecasts with actual values.

Finally, the specification for the weekly U.S. Gasoline Data of De Livera et al. (2011) consists of equations (41)–(42) with a constant slope for the trend, and one seasonal component, $s_t = \sum_{i=1}^{m} s_{i,t}$, with non–integer seasonal period, $n = 365.25/7 = 52.179$, and a number of harmonics $m = 7$. Each harmonic, $s_{i,t}$, follows equations (53)–(54) with $\bar{k}_{i,1} = \bar{k}_1$ and for $\bar{k}_{i,2} = \bar{k}_2$. The stationary component, c_t, is assumed to follow an AR(1) model, $(1 + \phi B)c_t = k_3 a_t$. The last 261 observations were reserved to assess forecasting performance.

The estimated parameters for the three models are given in Table 1. The parameters were constrained to provide an invertible and stationary model.

In Figure 2, we can see the airline series of Box and Jenkins (1976) together with the three components given by the BN decomposition. All of the series correspond to the estimation sample. For the last 48 observations, the forecasts produced by the model and the actual values are also shown.

In Figure 3, the quarterly U.S. real GDP series of Oh et al. (2008), together with the trend and the cycle given by the BN decomposition, are shown. All of the series correspond to the estimation sample and, in the case of the cycle, the NBER expansion and contraction dates for the selected period are indicated. For the last 12 observations,

Table 1: Estimated parameters for each model based on the BN decomposition

	Parameters				
Data	k_1	k_2	$\bar{k}_1$	$\bar{k}_2$	ϕ
Airline	0.5082	0.0074	0.0398	0.0227	
Real GDP	1.2419	0.1933			
Gasoline	0.0500	0	0.0003	0.0008	0.2000

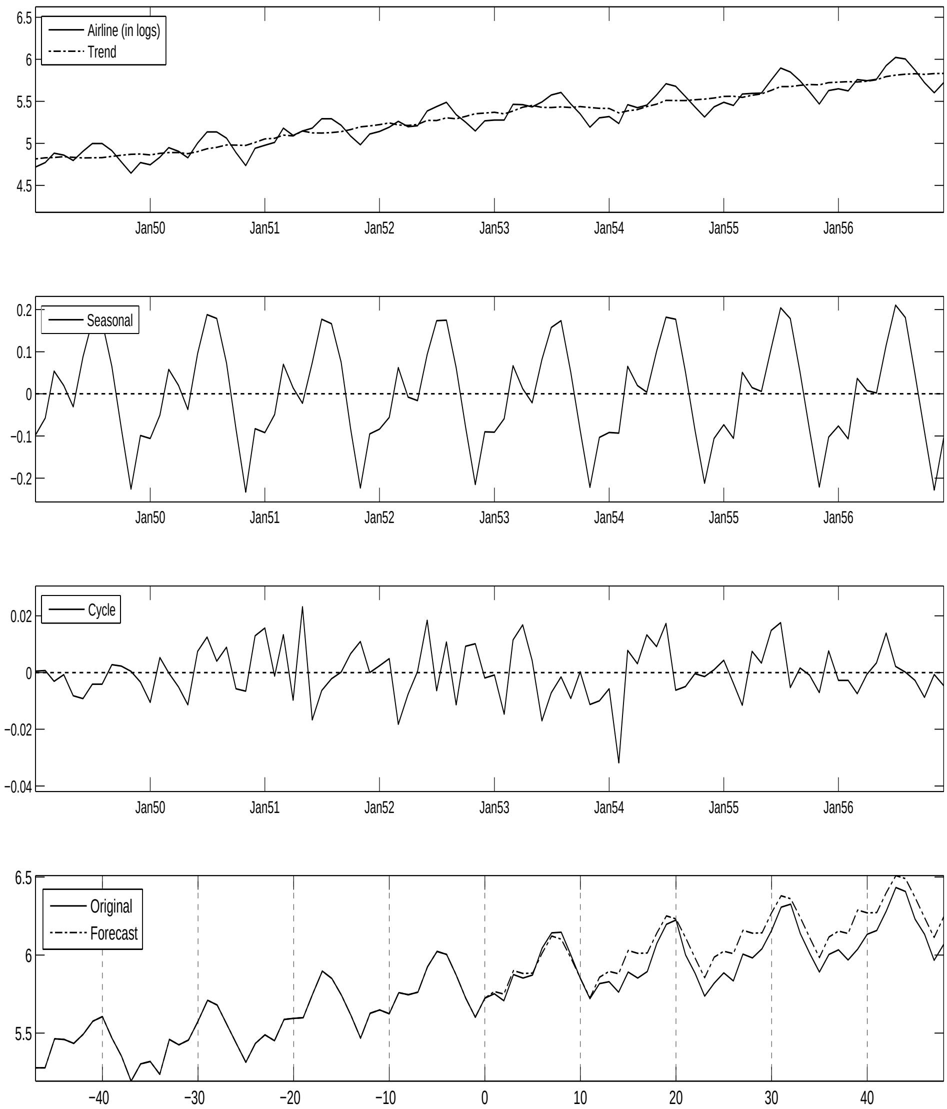

Figure 2: Airline Series in logs with its estimated BN components and 48 forecasts.

we can also see the forecasts produced by the model and the actual values.

Finally, in Figure 4, we can see the weekly U.S. Gasoline Data of De Livera et al. (2011) together with the three components given by the BN decomposition. All of the series correspond to the estimation sample. Also, for the last 261 observations, we can see both the forecasts produced by the model and the actual values.

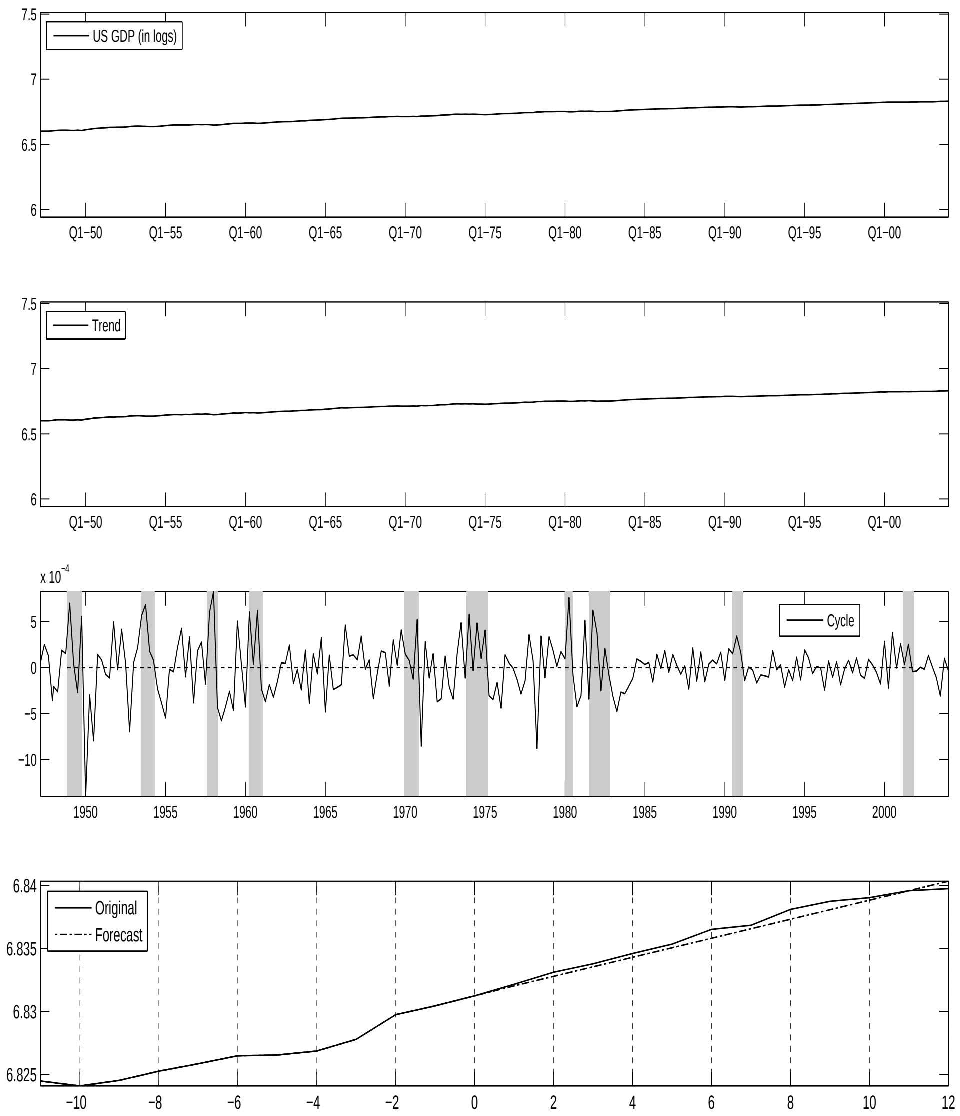

Figure 3: U.S. real GDP in logs with its estimated BN components and 12 forecasts. The NBER expansion and contraction dates are indicated for the cycle.

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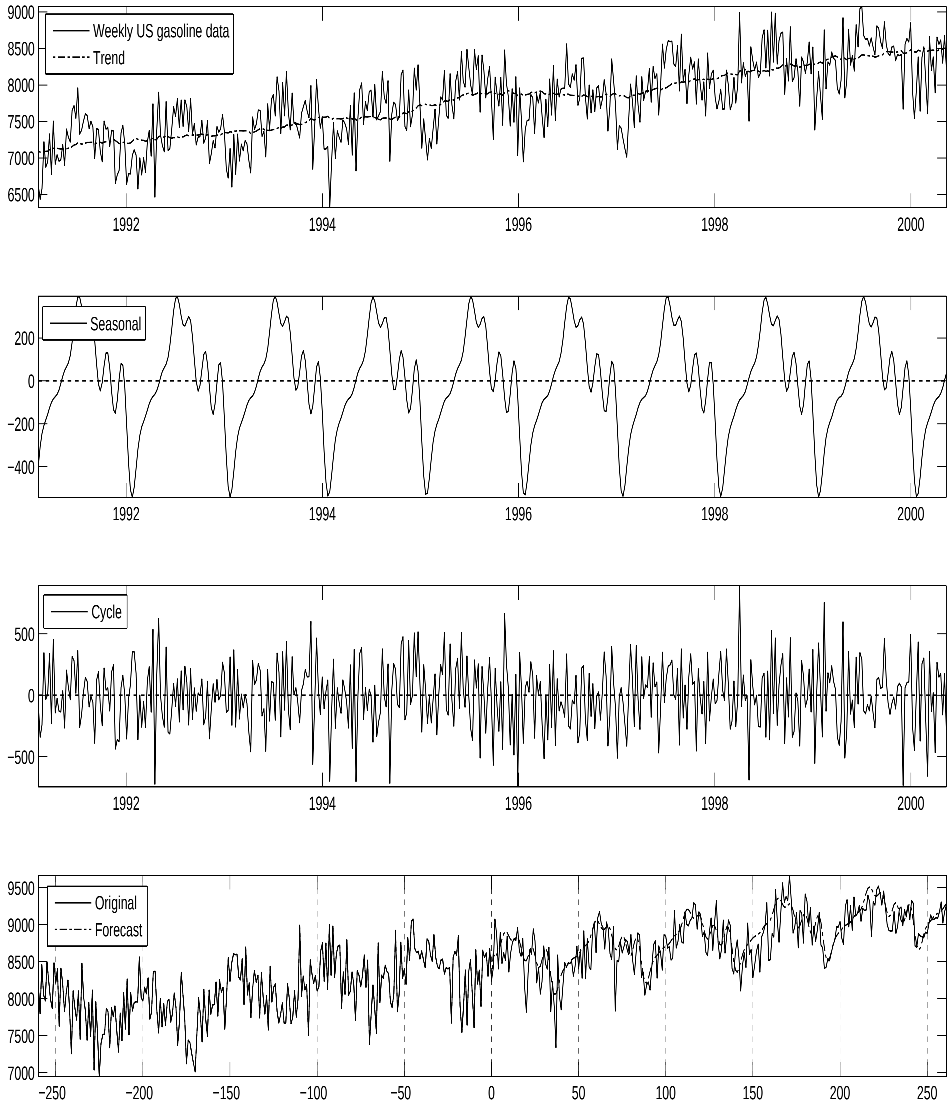

Figure 4: U.S. gasoline data with its estimated BN components and 261 forecasts.

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